Linear Transformations and Determinants

1. n-box:

Let $a_1, a_2, \ldots a_n$ be n independent vectors in R^m for $n \le m$. The n-box in R^m is determined by these vectors is the set of all vectors x satisfying the equation

$x = t_1 a_1 + t_2 a_2 + \ldots t_n a_n$
for $0 \le t_i \le 1$ and $i = 1, 2, \ldots n$.

Note: If $a_1, a_2, \ldots a_n$ are independent, then the set is a degenerate n-box.

2. Volume of a Box:

The volume of the n-box in R^m determined by independent vectors $a_1, a_2, \ldots a_n$ is given by

$$\text{volume} = \sqrt{|\det(A^T A)|}$$

Note: If A is a square matrix, then the volume is just det(A).

E.g. 1

Find the area of the parallelogram in R^4 determined by the vectors [2, 1, -1, 3] and [0, 2, 4, -1].

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ -1 & 4 \\ 3 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 15 & -5 \\ -5 & 21 \end{bmatrix}$$

Note: $A^T A$ is symmetric.

$$\det(A^T A) = (15)(21) - (-5)(-5)$$
$$= 290$$

$$\text{Area} = \sqrt{\det(A^T A)}$$
$$= \sqrt{290}$$

E.g. 2

Find the volume of the parallelepiped in $\mathbb{R}^3$ determined by the vectors $[1, 0, -1]$, $[-1, 1, 3]$ and $[2, 4, 1]$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ -1 & 3 & 1 \end{bmatrix}$$

Since A is a square matrix, the volume is equals to $|\det(A)|$.

$$|\det(A)| = 5$$

2. Volume Change Factor For $T: R^n \to R^n$:

Let G be a region in R^n with volume V, and let $T: R^n \to R^n$ be a linear transformation of rank n with standard matrix representation A. Then, the volume of the image of G under T is $|\det(A)| \cdot V$.

E.g. 3

Let $T: R^2 \to R^2$ be the linear transformation of the plane given by $T[x,y] = [2x-y, x+3y]$. Find the area of the image under T of the disk $x^2+y^2 \leq 4$ in the domain of T.

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} V &= \pi r^2 \\ &= 4\pi \end{aligned}$$

Recall the formula of a circle: $(x-h)^2 + (y-k)^2 \leq r^2$ where r is the radius.

$$\begin{aligned} &|\det(A)| \cdot V \\ &= |(2)(3)-(1)(-1)| \cdot 4\pi \\ &= 7 \cdot 4\pi \\ &= 28\pi \end{aligned}$$

Note: If A is not a square matrix, then the formula is volume $= \sqrt{|\det(A^T A)|} \cdot V$.

E.g. 4

Let $T: R^2 \to R^2$ be the lin trans defined by $T[x,y] = [4x-2y, 2x+3y]$. Find the area of the image under T of the rectangle $-1 \le x \le 1$, $1 \le y \le 2$ in the domain of T.

$$A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$$

$$V = (2)(1)$$
$$= 2$$

$$\text{Area} = |\det(A)| \cdot V$$
$$= |(4)(3) - (-2)(2)| \cdot 2$$
$$= 16 \cdot 2$$
$$= 32$$

E.g. 5

Find the area of the triangle in R^3 with vertices $(-1,2,3)$, $(0,1,4)$, and $(2,1,3)$.

Solution:

Choose any of the vertices and fix it. Then, subtract all the other vertices from it and use those in your matrix.

In this case, i'll fix $(-1,2,3)$.

$(0,1,4) - (-1,2,3) = [1,-1,1]$

$(2,1,3) - (-1,2,3) = [3,-1,2]$

$$A = \begin{bmatrix} 1 & 3 \\ -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\sqrt{|\det(A^T A)|} = \sqrt{6}$$

Now, because this is a triangle, the area is $\frac{\sqrt{6}}{2}$.

E.g. 6

Find the vol of the tetrahedron in R^4 with vertices $(1,0,0,1)$, $(-1,2,0,1)$, $(3,0,1,1)$ and $(-1,4,0,1)$.

Solution:

I'll fix $(1,0,0,1)$.

$(-1,2,0,1) - (1,0,0,1) = [-2,2,0,0]$

$(3,0,1,1) - (1,0,0,1) = [2,0,1,0]$

$(-1,4,0,1) - (1,0,0,1) = [-2,4,0,0]$

$$A = \begin{bmatrix} -2 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 4 & 0 & 0 \end{bmatrix}$$

$$\sqrt{|\det(A^T A)|} = 4$$

Since this is a tetrahedron, you divide your answer by 6.

$\therefore$ The vol is $\frac{4}{6}$ or $\frac{2}{3}$.

3. Co-Planar:

Suppose we have points a, b, c, and d. To see if they lie in the same plane in R^n, we need to check if they are co-planar. The points are co-planar iff $\det(A^T A) = 0$, where A is the matrix rep of the points.

E.g. 7

Determine whether or not the points $(1,0,1,0)$, $(-1,1,0,1)$, $(0,1,-1,1)$ and $(1,-1,4,-1)$ lie in a plane in R^4.

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$\det(A) = 0$, so the points are co-planar.

However, suppose we used these points instead. $(2,0,1,3)$, $(3,1,0,1)$, $(0,1,-1,1)$ and $(1,-1,4,-1)$.

$$A = \begin{bmatrix} 2 & 3 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 3 & 1 & 4 & 4 \end{bmatrix}$$

$\det(A) = -29$

$\neq 0$

$\therefore$ The points are not co-planar.